

M4041: Lecture 0

0. Multivariable Calculus

Notation 0.1 Let X, Y be finite dimensional vector spaces.

- (i) $L(X, Y)$ will denote the set of linear maps from X to Y . If $X = Y$, then we write $L(X) = L(X, X)$.
- (ii) If $U \subset X$ is open, then we let $C^\infty(U, Y)$ denote the set of smooth maps from U to Y .

Recall that the derivative Df of a smooth function $f \in C^\infty(U, Y)$ is defined by

$$Df(x) \cdot v := \frac{d}{dt} \Big|_{t=0} f(x+tv) \quad \forall x \in U, v \in X. \quad (0.1)$$

Note: When $U \subset \mathbb{R}$, we will often use the notation $f'(x)$ instead of $Df(x)$.

One important consequence of this definition is that

$$Df(x) \in L(X, Y) \quad \forall x \in U \quad (0.2)$$

Notation 0.2 Given an open set $U \subset X$ and $V \subset Y$, we will let

$$C^\infty(U, V) = \{f \in C^\infty(U, Y) \mid f(U) \subset V\}$$

Theorem 0.3 (Chain Rule) Let X, Y, Z be finite dimensional vector spaces and $U \subset X$, $V \subset Y$ open sets. If $f \in C^\infty(U, Z)$, $g \in C^\infty(V, Y)$ and $\text{Ran}(g) \subset V$, then $f \circ g \in C^\infty(U, Z)$ and

$$D(f \circ g)(x) = Df(g(x)) \circ Dg(x)$$

for all $x \in U$.

Let

$$X = \mathbb{R}^n, Y \in \mathbb{R}^m$$

and suppose that

$$f \in C^\infty(U, Y) \quad (U \subset X \text{ open})$$

so that

$$f(x) = (f^1(x), \dots, f^m(x)) \quad x = (x^1, \dots, x^n)$$

where

$$f^1, \dots, f^m \in C^\infty(U, Y).$$

Then for

$$v = (v^1, \dots, v^n) \in \mathbb{R}^n,$$

we have that

$$Df(x) \cdot v = \left. \frac{d}{dt} \right|_{t=0} f(x+tv)$$

$$= \left. \frac{d}{dt} \right|_{t=0} (f^1(x+tv), \dots, f^m(x+tv)) \quad (0.3)$$

But

$$\frac{d}{dt} \Big|_{t=0} f^j(x+tv) = \frac{d}{dt} \Big|_{t=0} f^j(x^1+tv^1, \dots, x^n+tv^n)$$

$$= D_i f^j(x) v^i \quad (0.4)$$

where

(by the chain rule)

$$D_i f^j(x) := \frac{d}{dt} \Big|_{t=0} f^j(x^1, \dots, x^{i-1}, x^i+t, x^{i+1}, \dots, x^n)$$

are the partial derivatives,
and we are employing the Einstein summation
convention where repeated indices are
summed over, i.e.

$$D_i f^j(x) v^i = \sum_{i=1}^n D_i f^j(x) v^i$$

Remark 0.4

(i) Unless otherwise stated, we will employ
the Einstein summation convention in this
unit.

(ii) We will use any of the following
notations for partial derivatives on \mathbb{R}^n :

$$D_i f(x) = \partial_i f(x) = \frac{\partial f(x)}{\partial x^i}$$

where

$$x = (x^1, \dots, x^n) \in \mathbb{R}^n$$

Using (0.3) and (0.4), we get the following matrix representation

$$Df(x) \cdot v = \begin{bmatrix} D_1 f^1(x) & D_2 f^1(x) & \cdots & D_n f^1(x) \\ D_1 f^2(x) & D_2 f^2(x) & \cdots & D_n f^2(x) \\ \vdots & \vdots & & \vdots \\ D_1 f^m(x) & D_2 f^m(x) & \cdots & D_n f^m(x) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v^n \end{bmatrix}, \quad (0.5)$$

or in components,

$$(Df(x) \cdot v)^j = D_j f^j(x) v^j \quad j=1, 2, \dots, m$$

Definition 0.5

If X_1, X_2, Y are finite dimensional vector spaces, $U_1 \subset X_1$ and $U_2 \subset X_2$ are open subsets, and $f \in C^\infty(U_1 \times U_2, Y)$, then

$$D_1 f(x_1, x_2) v_1 := \left. \frac{d}{dt} \right|_{t=0} f(x_1 + t v_1, x_2) \quad (x_1, x_2) \in U_1 \times U_2, v_1 \in X_1$$

and

$$D_2 f(x_1, x_2) v_2 := \left. \frac{d}{dt} \right|_{t=0} f(x_1, x_2 + t v_2) \quad (x_1, x_2) \in U_1 \times U_2, v_2 \in X_2$$

define the partial derivative operators with respect to the spaces X_1 and X_2 .

Theorem 0.6 (Inverse Function Theorem)

Suppose $U \subset X$ is open, $x_0 \in U$, and $f \in C^\infty(U, Y)$. If $Df(x_0)$ is invertible, then there exists an open neighborhood V_{y_0} of $y_0 = f(x_0)$ and a smooth map $g \in C^\infty(V_{y_0} \cap U)$ that satisfies

$$f(g(y)) = y \quad \forall y \in V_{y_0}.$$

Example 0.7

Consider

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (r, \theta) \mapsto (r\cos\theta, r\sin\theta).$$

Then

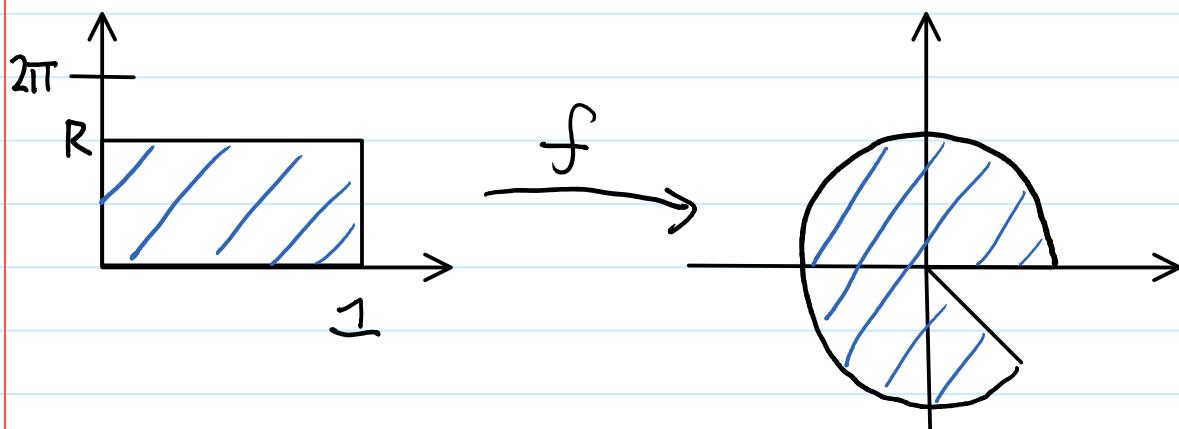
$$DF(r, \theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

and

$$\det DF(r, \theta) = r$$

Thus f is locally invertible for each point

$$(r, \theta) \in (0, \infty) \times (-\infty, \infty)$$



Theorem 0.8 (Implicit Function Theorem)

Suppose $U \subset X \times Y$ is open, $f \in C^\infty(U; Z)$,
and $(x_0, y_0) \in U$ satisfies

$$f(x_0, y_0) = 0$$

and

$$D_2 f(x_0, y_0) \in L(Y, Z) \text{ is invertible.}$$

Then there exists an open neighborhood V_{x_0} of x_0 in X , an open neighborhood W_{y_0} of y_0 in Y ,
and a $g \in C^\infty(V_{x_0}, W_{y_0})$ such that

$$f(x, g(x)) = 0 \quad \forall x \in V_{x_0}.$$

Moreover,

$$f(x, y) = 0 \quad \text{for } (x, y) \in V_{x_0} \times W_{y_0}$$

if and only if $y = g(x)$.

Example 0.9

Let

$$f(x,y) = e^{x+y} (x-2y)$$

Then

$$D_2 f(x,y) = e^{x+y} (x-2y-2)$$

and so,

$$f(2,1) = 0$$

and

$$D_2 f(2,1) = -2e^3 \neq 0.$$

Therefore, by the IFT, there exists an open interval I about 1 and a smooth function $g \in C^\infty(I)$ satisfying

$$f(x, g(x)) = 0 \quad \forall x \in I.$$